LIMITING DISTRIBUTIONS OF SUMS OF RATIOS OF SAMPLE SPACINGS WITH APPLICATIONS TO THE PARAMETRIC TWO TWO SAMPLE PROBLEM

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LIMITING DISTRIBUTIONS OF SUMS OF RATIOS OF SAMPLE SPACINGS WITH APPLICATIONS TO THE PARAMETRIC TWO SAMPLE PROBLEM.

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1. Introduction.

In this paper we are concerned with the distribution properties of certain statistics based on ratios of sample spacings from different populations. Sufficient conditions for the existence of limiting distributions are obtained and the forms of the distributions are given. Applications of these statistics to a "parametric" two sample hypothesis is also discussed.

Let $X_1,\ldots,X_m$ and $Y_1,\ldots,Y_n$ be a set of $(n+m)$ independent random variables, the first m having common c.d.f. $F(x)$ and the second n having common c.d.f. $G(x)$. Denote the two sets of ordered observations by $X_1' \leq\ldots\leq X_m'$ and $Y_1' \leq\ldots\leq Y_n'$. The sample spacings, or sample successive differences, from the two sets of random variables are given as

$$(1.1)\qquad \begin{aligned} DX_i &= X'_{i+1} - X'_i \quad , \quad i=1,\ldots,m-1 \\ DY_j &= Y'_{j+1} - Y'_j \quad , \quad j=1,\ldots,n-1 \,. \end{aligned}$$

If m=n, we can define the statistic

$$(1.2)\qquad S_n(r) = \sum_{i=1}^{n-1} (DX_i/DY_i)^r \qquad 0 < r \leq 1.$$

If $m > n$, a subset $X_{i_1},\ldots,X_{i_n}$ of $X_1,\ldots,X_m$ can be chosen at random and then $S_n(r)$ can be defined as above. We shall assume m=n whenever we discuss $S_n(r)$. Also of interest will be the "reciprocal" statistic

$$(1.3)\qquad S_n^{-1}(r) = \sum_{i=1}^{n-1} (DY_i/DX_i)^r \qquad 0 < r \leq 1.$$

Under certain assumptions about the behavior of $F(x)$ and $G(x)$ in the tails, limiting distributions are found for $S_n(r)$ in Section 3. It is found that as r varies from 0 to 1, the limiting distribution varies over the class of stable distributions with parameter α going from 1 to 2 (see Section 3). Stochastic convergence of $S_n(r)$ to a limit is taken up in Section 5. The results of Sections 3 and 5 are used in Section 6 where tests based on $S_n(r)$ are proposed for the "parametric" two sample problem, namely

$$(1.4)\qquad H'_o: F(x) = G((x-\mu)/\sigma)$$

where μ (real) and σ (> 0) are two unspecified "nuisance" parameters. It is shown that the statistic

$$S_n(r)\, S_n^{-1}(r) \qquad \tfrac{1}{2} \leq r \leq 1$$

is asymptotically distribution free under H_o', yields a consistent test and one whose size can be controlled approximately by means of the limiting distributions and whose power can be found by the same means. No attempt is made to compare the various tests as r ranges from $\frac{1}{2}$ to 1 because of the fact that only the characteristic functions of the limiting distributions are known.

In Section 6 we consider also the use of $S_n(r)$ to estimate σ when H_o' is known to be true. Also in that section we consider the statistic

$$(1.5) \qquad \prod(m, n) = \left(\prod_{i=1}^{m-1} DX_i\right)^{(1/m)} \Big/ \left(\prod_{j=1}^{n-1} DY_j\right)^{(1/n)}$$

as an estimator for σ. The results of Sections 4 and 5 are used in studying $\prod(m, n)$. Note that we need not have m=n to use this estimator.

In Section 7 we consider tests of the "parametric" goodness of fit problem where $G(x)$ is now specified and only one set of observations is made. Statistics analogous to $S_n(r)$ are described and a statistic based on the logarithms of the DX_i is described. This last is compared briefly with one proposed by Weiss [13] for this problem. Estimation of σ is considered also for this case.

We consider also the properties of and uses of truncated versions of $S_n(r)$, namely

$$S_n(r, u, v) = \sum_{i=[nu]}^{[nv]} (DX_i/DY_i)^r \qquad 0 < r \leq 1 \tag{1.6}$$

where $0 < u < v < 1$ are given and $[x]$ is the greatest integer not exceeding x. $S_n^{-1}(r, u, v)$ is defined analogously. To obtain limiting properties for $S_n(r, u, v)$ it is not necessary to put restrictions on the tail behavior of $F(x)$ and $G(x)$ since this statistic will in the limit not be affected by these tail properties.

In the following Section our notation and assumptions are detailed.

2. Preliminary Remarks.

In this Section we introduce the basic tools which are used in Sections 3, 4 and 5. The results of those sections depend on being able to express the set of random variables $\{DX_i/DY_i\}$, $i=1,\ldots,n-1$ in terms of a set of independent random variables. To establish this equivalence, the ratio (DX_i/DY_i) will be expanded in a Taylor series as a function of the hazard rate, and certain well known properties of the hazard rate will be used to make the connection with the set of independent random variables. This technique was used by the author in [3] and by Proschan and Pyke in [11] to obtain central limit theorems for functions of a single set of sample spacings. Renyi [12] used similar methods to study functions of order statistics.

We now define the hazard rate function and give some of its properties. For a more comprehensive discussion see Barlow, Marshall and Proschan [1]. Let $F(x)$ be a distribution function with density $f(x)$. The hazard rate $h(x)$ is defined

$$(2.1) \qquad h(x) = f(x)/(1-F(x)) .$$

The cumulative hazard rate $H(x)$ is

$$(2.2) \qquad H(x) = \int_{-\infty}^{x} h(t)dt = -\log(1-F(x)) .$$

It is easily seen that the random variable $H(X)$ (where X has c.d.f. $F(x)$) has the standard exponential distribution,

$$(2.3) \qquad P(H(X) \leq x) = 1-e^{-x} .$$

Since $H(x)$ is an increasing function of x, if $X_1' \leq \ldots \leq X_m'$ represent the ordered values of m independent random variables with common distribution $F(x)$, then $H(X_1') \leq \ldots \leq H(X_m')$ represent the ordered values of m independent exponentially distributed random variables. Let $U_1, U_2, \ldots, U_m$ be a set of m independent exponentially distributed random variables. When two random variables X and Y (say) have the same distribution we shall write $X \overset{\mathcal{L}}{=} Y$. The fundamental relation for this paper is that

$$(2.4) \qquad H(X_i') \overset{\mathcal{L}}{=} \sum_{j=1}^{i} (U_j/(n-j+1)) \qquad i=1,\ldots,m .$$

Epstein and Sobel first announced (2.4) in [5] and Renyi (see [12]) has pioneered in its application to the study of order statistics.

The spacing DX_i is related to the above through the expansions

$$(2.5) \qquad DX_i = (H(X_{i+1}')-H(X_i'))/h(\bar{X}_i) \; ; \quad X_i' \le \bar{X}_i \le X_{i+1}' \qquad i=1,\ldots,m-1$$

and

$$(2.6) \qquad DX_i = [(H(X_{i+1}')-H(X_i'))/h(i)][1-(H(X_{i+1}')-H(i))(h'(i)/h^2(i))] + \text{error term}.$$

We use the prime notation for derivatives, i.e., $h'(x) = dh(x)/dx$. By $h(i)$ we mean $h(x)$ evaluated at $x=F^{-1}(i/n)$. Similarly for $H(i)$ and $h'(i)$. The error term will be assumed to be negligible and will be ignored in all computations. This will be valid under mild regularity conditions on $h'(x)$. By $F^{-1}(x)$ is meant the usual inverse of the distribution function which is assumed to be unique for the values of x for which it is used. In fact we shall assume that $f(x) > 0$

over some open interval $(F^{-1}(u), F^{-1}(v))$, $0 \leq u < v \leq 1$, where u and v will be made apparent from the context.

Another useful expansion given to the second order is

$$(2.7) \qquad (1/DX_i) = [h(i)/H(X'_{i+1})-H(X'_i)][1+(H(X'_i)-H(i))(h'(i)/h^2(i))] .$$

We shall denote the hazard rate for $G(x)$ by $q(x)$, its derivative by $q'(x)$ and its integral by $Q(x)$. Further $q(i)$ shall mean $q(x)$ evaluated at $x = G^{-1}(i/n)$. The same assumptions apply to $G(x)$ as to $F(x)$. The above relations for $H(X'_i)$ apply also to $Q(Y'_j)$ and expansions similar to (2.5), (2.6), and (2.7) hold for DY_j and $(1/DY_j)$. We shall denote a set of n independent exponential random variables by $V_1,\ldots,V_n$. The V's and U's are taken to be independent.

We summarize the above as

<u>Lemma 2.1.</u>

Let $F(x)$ and $G(x)$ satisfy the above assumptions. Let $(U_1,\ldots,U_m, V_1,\ldots,V_n)$ be (m+n) independent exponentially distributed random variables. The following distribution equivalences hold up to a second order approximation.

$$(2.8) \qquad DX_i \overset{\mathscr{L}}{=} [U_{i+1}/(n-i)h(i)][1-(\sum_{k=1}^{i} (U_k/n-k+1)-H(i))(h'(i)/h^2(i))]$$

$$i=1,\ldots,m-1$$

$$(2.9) \qquad (1/DX_i) \overset{\mathscr{L}}{=} [(n-i)h(i)/U_{i+1}][1+(\sum_{k=1}^{i} (U_k/n-k+1)-H(i))(h'(i)/h^2(i))]$$

(2.10) $$DY_j \overset{\mathcal{L}}{=} [V_{j+1}/(n-j)q(j)][1-(\sum_{k=1}^{j}(V_k/n-k+1)-Q(j))(q'(j)/q^2(j))]$$

$j=1,\ldots,n-1$

(2.11) $$(1/DY_j) \overset{\mathcal{L}}{=} [(n-j)q(j)/V_{j+1}][1+(\sum_{k=1}^{j}(V_k/n-k+1)-Q(j))(q'(j)/q^2(j))] .$$

In studying convergence and distribution properties under the assumption that $F(x) = G((x-\mu)/\sigma)$, a simplification can be obtained by noting that

(2.12) $$DX_i \overset{\mathcal{L}}{=} \sigma DX_{1i} \qquad i=1,\ldots,m-1$$

where the random variables $X_{1i},\ldots,X_{1m}$ are independent with common distribution $G(x)$. Thus any theorem which is true when $F(x) = G(x)$ holds also when $F(x) = G((x-\mu)/\sigma)$ with the obvious substitution of (DX_i/σ) for DX_i.

For later reference we state two simple facts

(2.13) $$P((U_i/V_i)^r < x) = x^{(1/r)}/(1+x^{(1/r)}) \qquad r > 0$$

and

(2.14) $$E[(U_i/V_i)^r] = \int_0^\infty (x^r/(1+x)^2)dx = \pi r/\sin\pi r \qquad 0 < r < 1 .$$

3. Limiting Distributions for Ratios.

In this section we shall obtain first the limiting distributions for the sums $S_n(r) = \Sigma(DX_i/DY_i)^r$. (See (1.2).) We take the index of summation to run from 1 through n-1 throughout this and the following sections unless stated otherwise. We assume m=n. We shall obtain also limiting distributions for the products $S_n(r)\, S_n^{-1}(r)$. The results are based on a lemma establishing the connection between these sums and sums of the independent random variables discussed in Section 2. It is then possible to use standard limit theorems found in Gnedenko and Kolmogorov [8] to establish the forms of the limiting distributions of the sums of independent random variables. Except in the case where the limiting distribution is Normal, our results give the value of the characteristic function of the limiting distribution. Of course this identifies the distribution uniquely but without an expression for the distribution function itself, it is not possible to compute percentage points, etc.

Two types of distributions arise below. In Theorems 3.1 and 3.3, we obtain the so-called stable distributions. The characteristic function $\varphi(t)$ of these distributions can be represented by (see [8], p. 164)

$$\log \varphi(t) = i\eta t - c|t|^{\alpha}\{1+i\beta(t/|t|)\omega(t, \alpha)\} \tag{3.1}$$

where α, β, η, c are constants (η is any real number, $-1 \leqq \beta \leqq 1$, $0 < \alpha \leqq 2$, $c \geqq 0$) and

$$\omega(t, \alpha) = \begin{cases} \tan \pi\alpha/2 & \text{if } \alpha \neq 1 \\ (2/\pi)\tan|t| & \text{if } \alpha = 1 . \end{cases}$$

In Theorems 3.1 and 3.3, in each case we have $1 \leq \alpha \leq 2$. For $\alpha=2$, (3.1) represents the Normal distribution and we shall arrange the result so that the limiting distribution is the standard normal (mean 0, variance unity). We denote this distribution by $\Phi(x)$. For $1 \leq \alpha < 2$, we denote the distribution (3.1) by $S(\alpha, \beta, \eta, c)$ listing the values of these constants in the order given.

Each of the distributions in Theorems 3.1 and 3.3 has $|\beta| = 1$ and is skewed toward the right. Such distributions have been studied by Mandelbrot [10] in connection with Economic Theory and he has labelled them "Pareto-Levy" distributions (for $1 < \alpha < 2$). He also mentions some tabulation now in progress which when published will make the present results of more immediate usefulness. For discussions of stable distributions in general the reader is referred to Lukacs [9] and to Fisz [6].

In Theorem 3.2, we obtain as limiting distributions infinitely divisible (I. D.) distributions of the class L which arises from limits of sums of independent random variables. These can be represented in general by

$$\log \varphi(t) = i\eta t-(\sigma^2 t^2/2)+ \int_{-\infty}^{0} (e^{iut}-1- \frac{iut}{1+u^2})dM(u)+ \int_{0}^{\infty} (e^{iut}-1- \frac{iut}{1+u^2})dN(u) \tag{3.2}$$

where η is any constant, $\sigma^2 \geqq 0$, and $M(u)$, $N(u)$ satisfy certain regularity conditions (see [8], p. 84). We shall have $\sigma^2=0$ in each case in Theorem 3.2

and $M(u) = 0$ (all u). Also we shall have $(-N(u)) = x^{-a}$ in each case. Thus we shall indicate the distributions (3.2) simply as $L(\eta, a)$ in the theorem.

If the distributions of a sequence of random variables X_n approach a distribution $F(x)$, we shall write $X_n \xrightarrow{\mathcal{L}} F(x)$. If a sequence of random variables X_n approaches stochastically a constant, random variable, or another sequence of random variables--denoted Y_n--we write $X_n \xrightarrow{P} Y_n$. If the sequence X_n has in the limit the same distribution as Y_n, we write $X_n \xrightarrow{\mathcal{L}} Y_n$.

Lemma 3.1.

Let $F(x)$ and $G(x)$ be distributions such that the hazard rates $h(x)$ and $q(x)$ satisfy the following conditions:

There is a constant $A > 0$, such that

(3.3) $$(1/A) < q(i)/h(i) < A .$$

$(h'(x)/h^2(x))$ and $(q'(x)/q^2(x))$ are both continuous and bounded functions of x.

Then as n increases,

(3.4a) $$(1/n^r)\Sigma(DX_i/DY_i)^r \xrightarrow{\mathcal{L}} (1/n^r)\Sigma(U_i/V_i)^r(q(i)/h(i))^r \qquad \tfrac{1}{2} < r \leq 1$$

(3.4b) $$(1/n \log n)^{\frac{1}{2}}\Sigma(DX_i/DY_i)^{\frac{1}{2}} \xrightarrow{\mathcal{L}} (1/n \log n)^{\frac{1}{2}}\Sigma(U_i/V_i)^{\frac{1}{2}}(q(i)/h(i))^{\frac{1}{2}}$$

$$0 < r < \tfrac{1}{2}$$

(3.4c) $$(1/n)^{\frac{1}{2}}\Sigma(DX_i/DY_i)^r \xrightarrow{\mathcal{L}} (1/n)^{\frac{1}{2}}[\Sigma(U_i/V_i)^r(q(i)/h(i))^r + \frac{(\pi r^2(U_i-V_i)}{(\sin\pi r)(n-i+1)} \sum_{j=i+1}^{n} (q^r(j)h'(j)/h^{2+r}(j))] .$$

Proof.

First we examine (3.4a) for $r=1$. Using (2.8) and (2.11) we have

$$(3.5) \qquad (1/n)\Sigma(DX_i/DY_i)$$

$$= (1/n)\Sigma(q(i)/h(i))(U_{i+1}/V_{i+1})[1+(q'(i)/q^2(i))(\sum_{j=1}^{i}(V_j/(n-j+1))-Q(i))$$

$$-(h'(i)/h^2(i))(\sum_{j=1}^{i}(U_j/(n-j+1))-H(i))] + \text{error term} .$$

From (2.2), we note that

$$(3.6) \qquad H(i) = Q(i) = -\log(1-\frac{i}{n}) = \sum_{\theta=1}^{i} 1/(n-j+1) + \Delta_n/(n-i+1)$$

where $|\Delta_n| < B$. Using (3.6) in (3.5), we obtain

$$(3.7) \qquad (1/n)\Sigma(DX_i/DY_i)$$

$$\overset{\mathcal{L}}{=} (1/n)\Sigma(q(i)/h(i))(U_{i+1}/V_{i+1})[1+(q'(i)/q^2(i))(\sum_{j=1}^{i}(V_j-1)/(n-j+1))$$

$$-(h'(i)/h^2(i))(\sum_{j=1}^{i}(U_j-1)/(n-j+1))] + \text{smaller terms} .$$

It will now be shown that the right side of (3.7) converges stochastically to the right side of (3.4a). To do this, we demonstrate that all terms of (3.7) except the first converge stochastically to zero. It suffices to display the argument showing

$$(3.8) \qquad (1/n)\Sigma(q(i)h'(i)/h^3(i))(U_{i+1}/V_{i+1}) \sum_{j=1}^{i}(U_i-1)/(n-j+1) \xrightarrow{P} 0$$

as n increases.

It will be convenient to rewrite the left side of (3.8) in the form

$$(3.9)\qquad (1/n)\sum_{j=1}^{n-1}((U_j-1)/(n-j+1))\sum_{i=j+1}^{n}(q(i-1)h'(i-1)/h^3(i-1))(U_i/V_i)\ .$$

Setting r=1 in (2.13), it will be observed that the random variable (U_i/V_i) has no expectation, making a direct proof of (3.8) somewhat cumbersome. We introduce the "truncated" variables U_i', V_i' defined by

$$(3.10)\qquad \begin{aligned} U_i' &= U_i && \text{if } (U_i/V_i) \le n\log n \\ &= (n\log n)V_i && \text{if } (U_i/V_i) > n\log n \\ V_i' &= V_i && \text{if } (U_i/V_i) \ge 1/n\log n \\ &= (n\log n)U_i && \text{if } (U_i/V_i) < 1/n\log n\ . \end{aligned}$$

It is easily seen that

$$(3.11)\qquad \begin{aligned} E(U_i'-1) &= 1/(1+n\log n) \\ E(U_i'-1)^2 &= (1+(n\log n)^2)/(1+n\log n)^2 < 1 \end{aligned}$$

and that

$$E(U_i'/V_i') \le k_1\log(n\log n)$$

and

$$E(U_i'/V_i')^2 \le k_2\, n\log n$$

where k_1 and k_2 are appropriate constants.

It is also easy to verify that

$$(3.13) \qquad \sum_{i=1}^{n} P[(U_i'/V_i') \neq (U_i/V_i)] = n/(1+ n \log n)$$

which goes to zero as n increases. Similar expressions obtain for $\Sigma\, P(U_i' \neq U_i)$ and $\Sigma\, P(V_i' \neq V_i)$. Thus it is possible to substitute U_i' for U_i and V_i' for V_i in (3.9) and to claim that the stochastic convergence or divergence of the resultant expression

$$(3.14) \qquad (1/n) \sum_{j=1}^{n-1} ((U_j'-1)/(n-j+1)) \sum_{i=j+1}^{n} (q(i-1)h'(i-1)/h^3(i-1))(U_i'/V_i')$$

implies that of (3.9). Due to the finiteness of the moments of the terms in (3.14), direct computation of the expected square of (3.14) shows that this mean square converges to zero with increasing n provided that the quantity $(q(i)/h(i))(h'(i)/h^2(i))$ has a well behaved integral. If $(q(i)/h(i))$ is bounded, it is necessary that both integrals

$$\int_0^1 (1/(1-y))^2 dy \left(\int_{F^{-1}(y)}^{\infty} (h'(x)f(x)/h^2(x))dx \right)^2$$

and

$$\int_0^1 (1/(1-y)^2)dy \int_{F^{-1}(y)}^{\infty} (h'(x)/h^2(x))^2 f(x)dx$$

exist. This will be so if $(h'(x)/h^2(x))$ is bounded for instance. In terms of the tail behavior of the distribution $F(x)$, $|h'(x)/h^2(x)| < B$ implies

$$(1-F(x))^{B+1} < f(x) < (1-F(x))^{-B+1} .$$

It also implies that $xf(x)/(1-F(x))$ (i.e., $xh(x)$) is bounded away from zero. For an example of an $F(x)$ which violates all of these conditions, we can consider $(1-F(x)) = e^{-(1/x)}$ for large x. For this distribution, (3.8) will not be true.

We have been considering only the situation where $(q(i)/h(i))$ is bounded which is somewhat of a restriction. It does include the important case $F(x) = G(x)$ and includes enough pairs $F(x) \neq G(x)$ so that the result should be useful in obtaining expressions for asymptotic power in a "neighborhood" of H_o' in testing the H_o' discussed in Section 1.

This completes the demonstration of (3.4a) for r=1. There is no essential difference for $(\frac{1}{2}) < r < 1$, nor for (3.4b). In examining (3.4c) it is not necessary to use the U_i' and V_i' since for $r < \frac{1}{2}$, $E(U_i/V_i)^{2r}$ exists. One does however have to be careful in computing the magnitude of the terms in the Taylor expansion as seen by the fact that two of them remain in the final expression. The details being similar to those used by Proschan and Pyke in [10] and by Blumenthal in [3], will be omitted. This completes the proof.

It will be observed that Lemma 3.1 would be true with the obvious interchange of U's and V's and q's and h's if DX_i/DY_i were replaced by DY_i/DX_i.

The limiting distributions of the quantities on the right sides of (3.4) can be determined by use of theorems in Gnedenko and Kolmogorov [8].

Theorem 3.1.

If $F(x) = G(x)$ and $F(x)$ satisfies the conditions of Lemma 3.1, then as n increases,

(3.15a) $$\log n[(1/n \log n)\Sigma(DX_i/DY_i)-1] \xrightarrow{\mathcal{L}} S(1,\ 1,\ -1,\ \pi/2)$$

(3.15b) $$(1/n^r)\Sigma[(DX_i/DY_i)^r-(\pi r/\sin\pi r)] \xrightarrow{\mathcal{L}} S[(1/r),\ -1,\ 0,\ (-(1/r)M(1/r)\cos(\pi/2r))]$$

$$\text{where } M(1/r) = \int_0^\infty [(e^{-y}-1+y)/y^{1+(1/r)}]dy$$

(3.15c) $$(2/n \log n)^{\frac{1}{2}}\Sigma[(DX_i/DY_i)^{\frac{1}{2}}-\pi/2] \xrightarrow{\mathcal{L}} \Phi(x)$$

(3.15d) $$(1/n\sigma_o^2(n,\ r))^{\frac{1}{2}}\Sigma[(DX_i/DY_i)^r - \pi r/\sin\pi r] \xrightarrow{\mathcal{L}} \Phi(x)$$

$$\text{where } \sigma_o^2(n,\ r) = \left[\frac{2\pi r}{\sin 2\pi r} - \frac{(\pi r)^2}{\sin^2\pi r}\right] + \left(\frac{2\pi r^2}{\sin\pi r}\right)^2\Sigma\,\frac{1}{n-i+1}\sum_{j=i+1}^{n}(h'(j)/h^2(j))$$

$$+ \frac{2(\pi r^2)^2}{\sin^2\pi r}\Sigma\left[\frac{1}{n-i+1}\sum_{j=i+1}^{n}(h'(j)/h^2(j))\right]^2 .$$

Proof.

Lemma 3.1 allows application of the results in section 35 of Gnedenko and Kolmogorov [8] for sums of independent, identically distributed random variables. Use of Theorem 2 of section 35 yields both (3.15a) and (3.15b). The constants

α and β are found by using equation (15) of section 34 and equations (7) and (8) of section 35. The constant c follows from the definitions used in section 34. In (3.15a), η can be found by direct evaluation of the characteristic function of $[(1/n)\Sigma(U_i/V_i)-\log n]$. The sums in (3.15b) are in the "domain of normal attraction" of their limit laws.

From Theorem 1, section 35, (3.15c) follows. Notice that the sum here is not in the "domain of normal attraction" of the normal law since the second moment of $(U_i/V_i)^{\frac{1}{2}}$ does not exist.

In (3.15d), section 35 does not apply since the quantities on the right side of (3.4c) are not identically distributed. The limiting normality is easily verified by the standard central limit theorem (Theorem 4, section 21 of [8]). A direct computation of the variance of the right side of (3.4c) leads to the given value of $\sigma_o^2(n, r)$.

Next we consider the distributions when $F(x) \neq G(x)$.

Theorem 3.2.

If $F(x)$ and $G(x)$ satisfy the conditions of Lemma 3.1, the sums $A_n(r)$ converge to non zero limits, and $\sigma_1^2(n, r)$ converges to a non zero limit, then the distributions of the quantities below converge to the indicated I.D. laws.

$$\log n[(1/n \log n\, A_n(1))\Sigma(DX_i/DY_i)-1] \xrightarrow{\mathcal{L}} L(-1, 1) \tag{3.16a}$$

$$(1/n^r A_n(r))\Sigma[(DX_i/DY_i)^r-(\pi r q^r(i)/h^r(i)\sin\pi r)] \xrightarrow{\mathcal{L}} L(0, 1/r) \quad \tfrac{1}{2} < r < 1 \tag{3.16b}$$

$$(2/n \log n\, A_n(\tfrac{1}{2}))^{\frac{1}{2}}\Sigma[(DX_i/DY_i)^{\frac{1}{2}}-(\pi/2)(q(i)/h(i))^{\frac{1}{2}}] \xrightarrow{\mathcal{L}} \Phi(x) \tag{3.16c}$$

where $A_n(r) = (1/n)\Sigma(q(i)/h(i))^{1/r}$

$$(1/n\sigma_1^2(n, r))^{\frac{1}{2}}\Sigma[(DX_i/DY_i)^r-(\pi r q^r(i)/h^r(i)\sin\pi r)] \xrightarrow{\mathcal{L}} \Phi(x) \qquad 0 < r < \tfrac{1}{2} \tag{3.16d}$$

$$\text{where } \sigma_1^2(n, r) = (1/n)\Sigma\{(\frac{2\pi r}{\sin 2\pi r} - \frac{(\pi r)^2}{(\sin\pi r)^2})(q(i)/h(i))^{2r}$$

$$+ (\frac{2\pi r^2}{\sin\pi r})^2(q(i)/h(i))^r(\frac{1}{n-i+1})\sum_{j=i+1}^{n}\frac{h'(j)q^r(j)}{h^{2+r}(j)}$$

$$+ 2(\frac{\pi r^2}{(n+1-i)\sin\pi r}\sum_{j=i+1}^{n}\frac{h'(j)q^r(j)}{h^{2+r}(j)})^2\}$$

Proof.

Using Lemma 3.1, the problem is reduced to one concerning independent random variables. Equations (3.16a) and (3.16b) are verified using Theorem 1, section 25 of [8]. The fact that $\sigma^2=0$ follows from the convergence of $A_n(r)$ and equation (11), section 35 of [8] which holds by virtue of our Theorem 3.1.

Equation (3.16c) follows from Theorem 4, section 26 of [8]. Equation (3.16d) is a result of the standard central limit theorem.

Remark.

The preceding results are based on the smooth behavior of the tails of the distributions through the conditions put on the hazard rates. It is possible to compute "truncated statistics", i.e., $S_n(r, u, v)$ as given by (1.6). For these statistics, theorems similar to the above apply also, with the constants modified appropriately to account for the abbreviated range of summation. The conditions for the truncated sums to have limiting distributions involve only the "central behavior", i.e., only the behavior in $(F^{-1}(u), F^{-1}(v))$ and $(G^{-1}(u), G^{-1}(v))$ (which quantities are assumed to be unique). Thus the tails can misbehave badly and some results are still obtainable.

It should be clear that due to the symmetry of our assumptions, (DX_i/DY_i) can be replaced by (DY_i/DX_i) in Theorems 3.1 and 3.2 and they will still be true with $q(i)$ and $h(i)$ interchanged. We shall use this fact below.

In Section 1, the statistics $S_n(r)\, S_n^{-1}(r)$ were mentioned. We shall now study the limiting distributions of this product in the important special case, $F(x) = G(x)$. The general case can be handled by the same methods.

<u>Lemma 3.2.</u>

Let $F(x) = G(x)$ and the conditions of Lemma 3.1 be satisfied. Then,

(3.17a) $$\log n[(1/n \log n)^2\Sigma(DX_i/DY_i)\Sigma(DY_i/DX_i)-1]$$

$$\xrightarrow{\mathcal{L}} (1/n)\Sigma((U_i/V_i)+(V_i/U_i)-2\log n)$$

(3.17b) $$(n^{(1-r)}\pi r/\sin\pi r)[(\sin\pi r/n\pi r)^2\Sigma(DX_i/DY_i)^r\Sigma(DY_i/DX_i)^r-1]$$

$$\xrightarrow{\mathcal{L}} (1/n^r)\Sigma((U_i/V_i)^r+(V_i/U_i)^r-2(\pi r/\sin\pi r)) \qquad \tfrac{1}{2} < r < 1$$

(3.17c) $$(\pi n^{\frac{1}{2}}/2(\log n)^{\frac{1}{2}})[(2/\pi n)^2\Sigma(DX_i/DY_i)^{\frac{1}{2}}\Sigma(DY_i/DX_i)^{\frac{1}{2}}-1]$$

$$\xrightarrow{\mathcal{L}} (1/n \log n)^{\frac{1}{2}}\Sigma((U_i/V_i)^{\frac{1}{2}}+(V_i/U_i)^{\frac{1}{2}}-\pi)$$

(3.17d) $$(\pi r/\sin\pi r)(1/n^{\frac{1}{2}})[(\sin\pi r/n\pi r)^2\Sigma(DX_i/DY_i)^r\Sigma(DY_i/DX_i)^r-1]$$

$$\xrightarrow{\mathcal{L}} (1/n)^{\frac{1}{2}}\Sigma[(U_i/V_i)^r+(V_i/U_i)^r-(2\pi r/\sin\pi r)$$

$$+\frac{\pi r^2(U_i-V_i)}{(\sin\pi r)(n-i+1)}\sum_{j=i+1}^{n}(h'(j)/h^2(j))] \qquad 0 < r < \tfrac{1}{2} \ .$$

Proof.

We use Lemmas 3.1 and 5.1. We examine in detail only (3.17a). By the above lemmas, we have

$$\log n[(1/n\log n)^2\Sigma(DX_i/DY_i)\Sigma(DY_i/DX_i)-1]$$

$$\xrightarrow{\mathcal{L}} \log n[(1/n \log n)^2\Sigma(U_i/V_i)\Sigma(V_i/U_i)-1]$$

$$= (1/n)\Sigma((U_i/V_i)-\log n)+(1/n \log n)(\Sigma(U_i/V_i))[(1/n)\Sigma((V_i/U_i)-\log n)]$$

$$\xrightarrow{\mathcal{L}} (1/n)\Sigma((U_i/V_i)-\log n)+(1/n)\Sigma((V_i/U_i)-\log n) \ .$$

The last statement is a consequence of Slutsky's Theorem (Cramer [3a], p. 254).

This completes the proof.

The right sides of (3.17) are sums of independent random variables which can be handled as in Theorem 3.1. It is only necessary to observe that

$$(3.18) \quad P((U_i/V_i)^r+(V_i/U_i)^r < x)$$

$$= \begin{cases} 0 & \text{for } x < 2 \\ 2^{1/r}[(x+\sqrt{x^2-4})^{1/r}-(x-\sqrt{x^2-4})^{1/r}]/(2^{1/r}+(x-\sqrt{x^2-4})^{1/r})(2^{1/r}+(x+\sqrt{x^2-4})^{1/r}) & \text{for } x \geq 2 . \end{cases}$$

Then the same approach wihch yielded Theorem 3.1 yields the results which we summarize as

<u>Theorem 3.3.</u>

Let $F(x) = G(x)$ and the conditions of Lemma 3.1 be satisfied. Then as n increases

$$(3.19a) \quad \log n[(1/n \log n)^2\Sigma(DX_i/DY_i)\Sigma(DY_i/DX_i)-1] \xrightarrow{\mathcal{L}} S(1,\ 1,\ -2,\ \pi/2)$$

$$(3.19b) \quad (n^{(1-r)}\pi r/\sin\pi r)[(\sin\pi r/n\pi r)^2\Sigma(DX_i/DY_i)^r\Sigma(DY_i/DX_i)^r-1]$$

$$\xrightarrow{\mathcal{L}} S[1/r,\ -1,\ 0,\ (-(2/r)M(1/r)\cos(\pi/2r))]$$

where $M(1/r)$ is given in (3.15b) $\qquad \frac{1}{2} < r < 1$

$$(3.19c) \quad (\pi n^{\frac{1}{2}}/(2\log n)^{\frac{1}{2}})[(2/\pi n)^2\Sigma(DX_i/DY_i)^{\frac{1}{2}}\Sigma(DY_i/DX_i)^{\frac{1}{2}}-1] \xrightarrow{\mathcal{L}} \Phi(x)$$

$$(3.19d) \quad (\pi r/\sin\pi r)(1/n\sigma_2^2(n,\ r))^{\frac{1}{2}}[(\sin\pi r/n\pi r)^2\Sigma(DX_i/DY_i)^r\Sigma(DY_i/DX_i)^r-1]$$

$$\xrightarrow{\mathcal{L}} \Phi(x) \qquad 0 < r < \tfrac{1}{2}$$

where

$$\sigma_2^2(n, r) = \left(\frac{4\pi r}{\sin 2\pi r} - \frac{4\pi^2 r^2}{\sin^2 \pi r} + 2\right) + \frac{2\pi^2 r^4}{\sin^2 \pi r} \left(\frac{1}{n} \Sigma\left[\frac{1}{n-i+1} \sum_{j=i+1}^{n} (h'(j)/h^2(j))\right]^2\right) .$$

We might note that for purposes of computing approximate probabilities, the quantities $\sigma_0^2(n, r)$, $\sigma_1^2(n, r)$, $\sigma_2^2(n, r)$ and $A_n(r)$ can be replaced by their limiting values.

4. Limiting Distributions for Logarithms.

In Sections 6 and 7 use is made of the products $\prod(m, n)$ (see (1.5)). Here we consider the distribution of the logarithm of this product, i.e.,

$$(4.1) \qquad \log\{(\prod_{i=1}^{m-1} DX_i)^{1/m} \Big/ (\prod_{j=1}^{n-1} DY_j)^{1/n}\} = (1/m) \sum_{i=1}^{m-1} \log DX_i - (1/n) \sum_{j=1}^{n-1} \log DY_j .$$

It will be observed that we need study only one of the sums, say $(1/m)\Sigma \log DX_i$, since the other will have the same distribution properties and will be independent of the first one. From the distribution of the logarithm, the distribution of the original product can be obtained.

The methods used here are the same as in Section 3 and we shall not present details.

Lemma 4.1.

Let $F(x)$ represent the common c.d.f. of $X_1, \ldots, X_m$. Let $G(x)$ be a specified distribution function. Let $F(x)$ and $G(x)$ satisfy the conditions of Lemma 3.1. Then as n increases

$$(4.2a)\quad (1/m)^{\frac{1}{2}}\Sigma(\log DX_i+\log g(i)+\gamma) \xrightarrow{\mathcal{L}} (1/m)^{\frac{1}{2}}\Sigma[(\log U_i+\log(g(i)/f(i))+\gamma) - ((U_i-1)/(m-i+1)) \sum_{j=i+1}^{m} (h'(j)/h^2(j))]$$

$$(4.2b)\quad (1/m)^{\frac{1}{2}}\Sigma(\log DX_i+\log g(i)+\gamma)^2 \xrightarrow{\mathcal{L}} (1/m)^{\frac{1}{2}}\Sigma[(\log U_i+\log(g(i)/f(i))+\gamma)^2 - 2((U_i-1)/(m-i+1) \sum_{j=i+1}^{m} (h'(i)/h^2(i))(\log(g(i)/f(i)))]$$

where

$$\gamma = -\int_0^\infty \log x\, e^{-x}dx = 0.5772\ldots \qquad \text{(Euler's constant).}$$

We omit the proof.

Theorem 4.1.

Let $F(x)$ and $G(x)$ be as in Lemma 4.1. Then as n increases

$$(4.3a)\quad (1/m\ \tau_1)^{\frac{1}{2}}\Sigma[\log(g(i)DX_i) + \gamma - \int_0^1 \log(g(G^{-1}(x))/f(F^{-1}(x)))dx] \xrightarrow{\mathcal{L}} \Phi(x)$$

where

$$\tau_1 = (\delta-\gamma^2-1) + \int_{-\infty}^{\infty} f(x)\log^2 f(x)dx - (\int_{-\infty}^{\infty} f(x)\log f(x))^2,$$

$$\delta = \int_0^\infty \log^2 x\, e^{-x}dx \quad \text{and } \gamma \text{ is given in (4.2).}$$

$$(4.3b)\quad (1/m\ \tau_2)^{\frac{1}{2}}\Sigma[(\log(g(i)DX_i)+\gamma)^2 - (\delta-\gamma^2) - \int_0^1 \log^2(g(G^{-1}(x))/f(F^{-1}(x)))dx] \xrightarrow{\mathcal{L}} \Phi(x)$$

where

$$\tau_2 = (\delta''+4\gamma\delta'+8\gamma^2\delta-4\gamma^4) + 4(\delta'+3\gamma\delta-2\gamma^3) \int_0^1 \log(gG^{-1}(x)/fF^{-1}(x))dx$$

$$+ 4(\delta-\gamma^2) \int_0^1 \log^2(gG^{-1}(x)/fF^{-1}(x))dx$$

$$+ 4 \int_0^1 dy(\frac{1}{1-y} \int_y^1 [1+(1-x)(f'F^{-1}(x)/f^2F^{-1}(x))] \log(gG^{-1}(x)/fF^{-1}(x))dx)^2$$

$$- 8 \int_0^1 (dy/(1-y))\log(gG^{-1}(y)/fF^{-1}(y)) \int_y^1 [1+(1-x)(f'F^{-1}(x)/f^2F^{-1}(x))]$$

$$\log(gG^{-1}(x)/fF^{-1}(x))dx.$$

Further, the quantities on the left sides of (4.3a) and (4.3b) have asymptotically a joint normal distribution with means of 0, unit variances, and covariance τ_3 given by

$$(4.4) \quad \tau_3 = (1/\tau_1\tau_2)\{\delta'+\gamma(3\delta-2\gamma^2)+2(\delta-\gamma^2) \int_0^1 \log(gG^{-1}(x)/fF^{-1}(x))dx$$

$$- 2 \int_0^1 (dy/(1-y)) \int_y^1 [1+(1-x)(f'F^{-1}(x)/f^2F^{-1}(x)]\log(gG^{-1}(x)/fF^{-1}(x))dx$$

$$- 2 \int_0^1 (dy/(1-y))\log(gG^{-1}(x)/fF^{-1}(x)) \int_y^1 [1+(1-x)(f'F^{-1}(x)/f^2F^{-1}(x))]dx$$

$$+ 2 \int_0^1 dy(1/(1-y))^2(\int_y^1 [1+(1-x)(f'F^{-1}(x)/f^2F^{-1}(x))]dx)$$

$$(\int_y^1 [1+(1-u)(f'F^{-1}(u)/f^2F^{-1}(u))]\log(gG^{-1}(u)/fF^{-1}(u))du)\}.$$

We use

$$\delta' = \int_0^\infty \log^3 x\, e^{-x} dx, \quad \text{and} \quad \delta'' = \int_0^\infty \log^4 x\, e^{-x} dx.$$

Proof:

Both (4.3a) and (4.3b) follow from Lemma 4.1 and the central limit theorem. The values of τ_1 and τ_2 given are limiting values of the respective variances. It is easy to check that it is legitimate to use these quantities. The joint normality is easily verified by considering any linear combination of the two random variables involved, then applying in order the results of Lemma 4.1 and the central limit theorem to the linear combination. The argument showing the existence of a limiting joint distribution is the same as that used by Fraser in [7]. The value of τ_3 given is the limiting value. This completes the proof.

Note that when the specified distribution $G(x)$ and the true distribution $F(x)$ are the same, we have

$$\tau_2 = \delta''+4\gamma\delta'+8\gamma^2\delta-4\gamma^4$$

(4.5) and

$$\tau_3 = (1/\tau_1\tau_2)(\delta'+\gamma(3\delta-2\gamma^2)).$$

5. Stochastic Convergence.

In this section, we study the stochastic convergence of the statistics whose distributions were considered in the previous two sections. Results similar to these were obtained by the author in [2] under somewhat more restrictive conditions on the distribution functions. We shall take advantage of the distribution theory in obtaining the convergence theorems. Two types of conditions will be studied - the conditions of Lemma 3.1 under which convergence of the sum $S_n(r)$ $(0 < r \leq 1)$ can be shown (when it is normalized), and a weaker condition which restricts only the central portion of the distribution function and under which convergence of the truncated sums $S_n(r, u, v)$ can be obtained.

We shall also give the convergence properties of the sums of logarithms studied in Section 4.

Lemma 5.1.

Let $F(x)$ and $G(x)$ satisfy the conditions of Lemma 3.1. Then as n increases

(5.1a) $$(1/n \log n)\, \Sigma(DX_i/DY_i) \xrightarrow{\mathcal{L}} (1/n \log n)\, \Sigma(q(i)U_i/h(i)V_i)$$

(5.1b) $$(1/n)\, \Sigma(DX_i/DY_i)^r \xrightarrow{\mathcal{L}} (1/n)\, \Sigma(q(i)U_i/h(i)V_i)^r \qquad 0 < r < 1$$

(5.1c) $$(1/m)\, \Sigma \log DX_i \xrightarrow{\mathcal{L}} (1/m)\, \Sigma(\log U_i - \log f(i))$$

(5.1d) $$(1/m)\, \Sigma(\log DX_i + \log g(i) + \gamma)^2 \xrightarrow{\mathcal{L}} (1/m)\, \Sigma(\log U_i + \log(g(i)/f(i)) + \gamma)^2.$$

Proof:

(5.1a) and (5.1b) for $0 < r \leq \frac{1}{2}$ follow a-fortiori from Lemma 3.1. For $0 < r < \frac{1}{2}$, it is necessary to verify that

$$(5.2) \qquad (1/n)\,\Sigma((U_i - V_i)/(n-i+1)) \sum_{j=i+1}^{n} (q^r(j)h'(j)/h^{2+r}(j)) \xrightarrow{P} 0.$$

By taking the expectation of the square of this quantity and using the conditinns of the lemma, (5.2) is readily established. Essentially the same argument is used to derive (5.1c) and (5.1d) from Lemma 4.1. This completes the proof.

It is tempting to want to prove stochastic convergence under weaker conditions than those needed to establish a limiting distribution, but in this case it does not seem possible. The amount of weakening which could be done is greatly limited as is seen by the example cited in Lemma 3.1. If $F(x) = G(x) = 1-e^{-(1/x)}$, the relations (5.1) will be false since the distributions of the quantities on the left will equal the distributions of the quantities on the right plus a random variable with positive variance. The reader cancheck easily that (5.2) will not hold. Thus the relaxation of conditions could not be extensive.

Theorem 5.1.

Let $F(x)$ and $G(x)$ satisfy the conditions of Lemma 3.1. Then as n increases

$$(5.3a)\qquad (1/n \log n)\, \Sigma(DX_i/DY_i) \xrightarrow{P} \int_0^1 (g(G^{-1}(x))/f(F^{-1}(x)))dx$$

$$(5.3b)\qquad (1/n)\, \Sigma(DX_i/DY_i)^r \xrightarrow{P} (\pi r/\sin\pi r) \int_0^1 (g(G^{-1}(x))/f(F^{-1}(x)))^r dx \qquad 0 < r < 1$$

$$(5.3c)\qquad (1/m)\, \Sigma \log DX_i \xrightarrow{P} -\gamma + \int_0^1 \log f(F^{-1}(x))dx$$

$$(5.3d)\qquad (1/m)\, \Sigma(\log DX_i + \log g(i) + \gamma)^2 \xrightarrow{P} (\delta-\gamma^2) + \int_0^1 \log^2(g(G^{-1}(x))/f(F^{-1}(x)))dx$$

where γ is given in (4.2) and δ in (4.3).

Proof:

By Lemma 5.1, we need only consider the sums of independent random variables appearing in relations (5.1)

To establish (5.3a) or (5.3b) for $\frac{1}{2} \leq r < 1$, the Theorems of Section 28 of [8] can be used. The strong law of large numbers (Section 27 of [8]) establishes (5.3b) for $0 < r < \frac{1}{2}$, and both (5.3c) and (5.3d). In these latter cases, the functions of U_i and V_i converge in the strong sense of probability one to the given constants, but the convergences in Lemma 5.1 are not with probability one so that this statement can not be made in (5.3b), (5.3c) or (5.3d). The integrals given are the limiting values of the appropriate sums. (In (5.3a) and (5.3b) these are the quantities $A_n(r)$ which appear in Theorem 3.2.) This completes the proof.

We turn now to the truncated statistics $S_n(r, u, v)$. We assume that for some pair of numbers u_o, v_o $(0 \leq u_o < v_o \leq 1)$, $F^{-1}(u_o)$ and $F^{-1}(v_o)$ are uniquely determined and $F(x)$ has on the interval $[F^{-1}(u_o), F^{-1}(v_o)]$ a finite, positive derivative $f(x)$ which is continuous except possibly at a finite number of points. The same assumption is made for $G(x)$.

Theorem 5.2.

Under the above assumptions about $F(x)$ and $G(x)$, given any two numbers u, v satisfying

$$0 \leq u_o < u < v < v_o \leq 1$$

we have as n increases

$$(5.4a) \qquad (1/(v-u)\, n \log n) \sum_{i=[nu+1]}^{[nv-1]} (DX_i/DY_i) \xrightarrow{P} (1/(v-u)) \int_u^v (g(G^{-1}(x))/f(F^{-1}(x)))dx$$

$$(5.4b) \qquad (1/(v-u)n) \sum_{i=[nu+1]}^{[nv-1]} (DX_i/DY_i)^r \xrightarrow{P} (1/(v-u)) \int_u^v (g(G^{-1}(x))/f(F^{-1}(x)))^r dx$$

$$0 < r < 1$$

where $[x]$ is the greatest integer less than or equal to x.

Proof:

The given assumptions are used to establish convergence in law of the given random variables to the corresponding random variables in which U_i replaces DX_i, and V_i replaces DY_i. Application of the Theorems in Sections 27 and 28 of [8] to these latter random variables then yields the desired conclusions. The details, with a slightly different method of approach from the one used in this paper can be found in [2]. This completes the proof.

Note that similar truncation can be applied to the logarithmic functions with the same consequences.

6. Applications to the Two Sample Problem.

In this section, we shall consider some tests of H_o' (see (1.4)) and estimates of σ based on the sums $S_n(r)$.

In the following discussion, it will be assumed that $F(x)$ and $G(x)$ have connected supports and satisfy the conditions of Lemma 3.1.

Testing

Given two populations with c.d.f.'s $F(x)$ and $G(x)$ respectively, one generalization of the "two-sample" hypothesis H_o: $F(x) = G(x)$ is the "two-sample" hypothesis with nuisance location and scale parameters,

H_o': $F(x) = G((x-\mu)/\sigma)$ where the parameters μ (real) and σ (positive) are not specified. H_o' then admits the possibility of different measuring instruments being used in making the observations on the two populations, with and V_1 replaces DY_1. Application of the Theorems in Sections 27 and 28 of [8] to these latter random variables then yields the desired conclusions. The details, with a slightly different method of approach from the one used in this paper can be found in [8]. This completes the proof.

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H_o': $F(x) = G((x-\mu)/\sigma)$ where the parameters μ (real) and σ (positive) are not specified. H_o' then admits the possibility of different measuring instruments being used in making the observations on the two populations, with the relation between the zero's and scales being unknown. Darling [4] and other authors have referred to this as the "parametric" two sample problem.

The results of Sections 3 and 5 used in conjunction with Lemma 6.1 below which is due to Weiss [13], allow us to base tests of H_o' on the sums $S_n(r)$ and $S_n^{-1}(r)$.

The proposed test is: choose an r and compute the product $S_n(r)S_n^{-1}(r)$. If this is "too large," reject H_o'. We now examine the properties of this test. From Theorem 5.1 it is seen that as n increases

$$(6.1)\quad \begin{aligned} &(1/n \log n)^2 S_n(1)S_n^{-1}(1) \xrightarrow{P} \int_0^1 (gG^{-1}(x)/fF^{-1}(x))dx \int_0^1 (fF^{-1}(x)/gG^{-1}(x))dx \\ &(1/n)^2 S_n(r)S_n^{-1}(r) \xrightarrow{P} (\pi r/\sin\pi r)^2 \int_0^1 (gG^{-1}(x)/fF^{-1}(x))^r dx \int_0^1 (fF^{-1}(x)/gG^{-1}(x))^r dx \\ &\qquad 0 < r < 1. \end{aligned}$$

This equation is related to H_o' through the following.

<u>Lemma 6.1</u> (Weiss): Let $F(x)$ and $G(x)$ be two distribution functions and u, v $(0 \leq u < v \leq 1)$ be two given numbers, and assume $F^{-1}(u)$, $F^{-1}(v)$, $G^{-1}(u)$, $G^{-1}(v)$ are all uniquely determined. Let $F(x)$ have a derivative $f(x)$ on $[F^{-1}(u), F^{-1}(v)]$ and $G(x)$ have a derivative $g(x)$ on $[G^{-1}(u), G^{-1}(v)]$.

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The results of Sections 3 and 5 used in conjunction with Lemma 6.1 below which is due to Weiss [13], allow us to base tests of H_0' on the sums $S_n(r)$ and $S_n^{-1}(r)$.

The proposed test is: choose an r and compute the quotient $S_n(r)/S_n^{-1}(r)$. If this is "too large," reject H_0'. We now examine the properties of this test. From Theorem 5.1 it is seen that as n increases

$$
\begin{aligned}
&(1/n \log n)^2 S_n(1) S_n^{-1}(1) \xrightarrow{P} \int_0^1 \left(gF^{-1}(x)/fF^{-1}(x)\right) dx \int_0^1 \left(fG^{-1}(x)/gG^{-1}(x)\right) dx \\
&(1/n)^2 S_n(r) S_n^{-1}(r) \xrightarrow{P} (\pi r/\sin \pi r)^2 \int_0^1 \left(gF^{-1}(x)/fF^{-1}(x)\right)^r dx \int_0^1 \left(fG^{-1}(x)/gG^{-1}(x)\right)^r dx \\
&\qquad 0 < r < 1.
\end{aligned} \tag{6.1}
$$

This equation is related to H_0' through the following:

<u>Lemma 6.1</u> (Weiss): Let $F(x)$ and $G(x)$ be two distribution functions and u, v $(0 \leq u < v \leq 1)$ be two given numbers, and assume $F^{-1}(u)$, $F^{-1}(v)$, $G^{-1}(u)$, $G^{-1}(v)$ are all uniquely determined. Let $F(x)$ have a derivative $f(x)$ on $[F^{-1}(u), F^{-1}(v)]$ and $G(x)$ have a derivative $g(x)$ on $[G^{-1}(u), G^{-1}(v)]$.

If $F(x) = G((x-\mu)/\sigma)$ for all x in $[F^{-1}(u), F^{-1}(v)]$ for some constants μ, σ $(\sigma > 0)$, then $fF^{-1}(y) = (1/\sigma)\, gG^{-1}(y)$ for almost all y in $[u,v]$. If in addition, $f(x) > 0$ for x in $(F^{-1}(u), F^{-1}(v))$, the converse is true.

The proof is given in [13]. In [2], we give an example showing the necessity of the extra condition for the converse.

From the lemma, it is seen that when H_o' is true

$$(6.2)\quad \begin{cases} (1/n \log n)^2\, S_n(1) S_n^{-1}(1) \xrightarrow{P} 1 & \\ (\sin \pi r / n\pi r)^2\, S_n(r) S_n^{-1}(r) \xrightarrow{P} 1 & 0 < r < 1. \end{cases}$$

Also, it is easily seen that if a function $z(x)$ is not constant on the interval $[u,v]$, then

$$(6.3)\quad (1/v-u)^2 \int_u^v z(x)dx \int_u^v (1/z(x))dx \quad > 1.$$

By the converse to the lemma, when H_o' is not true, $(fF^{-1}(x)/gG^{-1}(x))$ will not be constant on $[0,1]$. This establishes that any test which rejects H_o' when $S_n(r)$ is "too large" will be consistent against all alternatives satisfying the restrictions mentioned above.

It is necessary to know what is meant by "$S_n(r)$ too large." An approximation for the critical region can be obtained by using the distribution theory of Theorem 3.3. It will be seen from (2.12) that this theorem is true if

H_o' is true. Thus critical regions can be found having a specified significance level α under H_o' if $(1/2) \leq r \leq 1$. Note that when $r < (1/2)$ the variance, thus the distribution of $S_n(r)$ depends on the function $F(x)$ which is assumed to be unknown, and no critical region can be computed. (If one could be computed, (6.2) and (6.3) indicate that the corresponding test would be consistent.) Analogous to Theorem 3.2, when H_o' is not true the limiting distribution of $S_n(r)$ can be found by the same methods and power computations could then be made (provided that the characteristic functions could be inverted). We summarize the above discussion in

Theorem 6.1 Let $F(x)$ and $G(x)$ have connected supports and satisfy the conditions of Lemma 3.1. Then a consistent test of H_o': $F(x) = G((x-\mu)/\sigma)$ is given by the rule: reject H_o' whenever $S_n(r)S_n^{-1}(r)$ is "too large," where $1/2 \leq r \leq 1$. A critical region of size approximately α can be found by using Theorem 3.3.

It is not clear what would happen if $F(x)$ and $G(x)$ do not have connected supports. If they do, but $(h'(x)/h^2(x))$ does not behave, the suggested test would have size different from α and would not be consistent since the normalized versions of $S_n(r)S_n^{-1}(r)$ would converge not to constants but to random variables (as was indicated after Lemma 5.1).

It is also possible to test a truncated or limited version of H_o', namely that $F(x) = G((x-\mu)/\sigma)$ for x in $(F^{-1}(u), F^{-1}(v))$ for some pair u, v $(0 < u < v < 1)$. Use of the appropriately truncated statistic $S_n(r,u,v)S_n^{-1}(r,u,v)$ can be made for this test. Consistency is demonstrated using Theorem 5.2 and distributions are easily computed as we remarked after Theorem 3.2. For the truncated statistic, the tail behavior of $F(x)$ is not important and the only condition needed for the existence of the limiting distributions is that $f(x) > 0$ over $(F^{-1}(u), F^{-1}(v))$ and similarly $g(x) > 0$ over $(G^{-1}(u), G^{-1}(v))$. Use of truncation at given percentiles of the distribution was made by Weiss in [13] to avoid trouble with the tails and also by Renyi in [12] for the same reason. The only alternative seems to be the use of restrictions such as those given here.

Estimation

Given that $F(x) = G((x-u)/\sigma)$ and the assumptions of Theorem 6.1 hold, we can estimate the "relative scale parameter" σ. To do this, any of the quantities

$$T_n(1) = (1/n \log n)\, S_n(1)$$

(6.4)

$$T_n(r) = [(\sin\pi r/\pi r n)\, S_n(r)]^{(1/r)} \qquad 0 < r < 1$$

can be used. Any one would be a consistent estimate by virtue of (2.12) and

Theorem 5.1 (with $F(x) = G(x)$). Using the distribution results of Theorem 3.1, it is possible to construct approximate confidence intervals for σ when $1/2 \leq r \leq 1$.

We note that $(1/(v-u))S_n(r,u,v)$ normalized as in (6.4) can be used to estimate σ if it is assumed only that $F(x) = G((x-\mu/\sigma)$ for x in $(F^{-1}(u), F^{-1}(v))$ for some u, v.

It is interesting to note that another estimate of σ is available through the products of the spacings $\pi(m,n)$ (see (1.5)). Note that (1.5) does not require $m = n$. Using Theorem 5.1 it is easily seen that $\pi(m,n)$ is a consistent estimate of σ and from Theorem 4.1 it is seen that it has a limiting log normal distribution with median σ.

It does not seem possible to construct a test of H_o' using products, (or alternatively logarithms) of sample spacings. A test analogous to (7.1) can be formed but will not be distribution free and so would not be of much use.

7. Applications to Tests of Fit.

The random variables $X_1,\dots,X_n$ are observed with common c.d.f. $F(x)$. The distribution $G(x)$ is specified and it is desired to test

$$H_o': \quad F(x) = G((x-\mu)/\sigma)$$

for some unspecified pair μ (real) and σ (> 0). The regularity conditions of Theorem 6.1 will be assumed.

In [13], Weiss considered a sample spacings test of this H_o'. Below, we shall suggest some others and compare one of them with the Weiss test. In [14], Weiss considered an estimator of σ. We shall suggest others below.

Imitating the approach of Section 6, we can test H_o' and estimate σ by using the analogs of $S_n(r)$ and $S_n^{-1}(r)$, namely

$$
\begin{aligned}
s_n(r) &= \sum_{i=1}^{n-1} (g(i)DX_i)^r \\
s_n^{-1}(r) &= \sum_{i=1}^{n-1} (g(i)DX_i)^{-r} \qquad 0 < r \leq 1.
\end{aligned}
\tag{7.1}
$$

Convergence and distribution theory for $s_n(r)$ and $s_n^{-1}(r)$ can be developed as for $S_n(r)$. Note that $s_n(r)$ and $s_n^{-1}(r)$ are not symmetric. For any $r > 0$, $s_n(r)$ will be asymptotically normal, while $s_n^{-1}(r)$ will have a stable distribution indexed by r as in Theorem 3.1. In considering the product $s_n(r)\, s_n^{-1}(r)$, the inverse dominates. For instance, when $r = 1$, and H_o' is true,

$$\log n \, [(1/n^2 \log n) \, s_n(1) \, s_n^{-1}(1) - 1]$$

has a limiting stable distribution with $\alpha = 1$. Since the limiting distributions for $1/2 < r \leq 1$ have no second moments, we would expect poor power in comparison with tests which are based on statistics with finite variances. On the other hand when $1/2 \leq r \leq 1$, the limiting distributions of $s_n(r)s_n^{-1}(r)$ under H_o' do not depend on $F(x)$ and are thus "distribution free." Note that $s_n(1/2) \, s_n^{-1}(1/2)$ has the added advantage of asymptotic normality. For $0 < r < 1/2$, the product is not distribution free. The same criticism can be made of Weiss's test statistic and of the one suggested below based on logarithms.

In Sections 4 and 5, the limiting distributions and convergence properties of sums of logarithms of sample spacings were studied. We shall now indicate how these functions can be applied to test H_o' and to estimate σ.

Consider the function

$$\frac{[(1/n) \, \Sigma(\log(g(i)DX_i)+\gamma)^2 - (\delta-\gamma^2)]^{\frac{1}{2}}}{|(1/n) \, \Sigma(\log(g(i)DX_i)+\gamma)|} \tag{7.2}$$

with absolute values in the denominator to insure its being positive. The constants γ and δ are given in (4.2) and (4.3) respectively.

By Theorem 5.1, the expression (7.2) converges stochastically to

$$(7.3) \qquad \frac{\left(\int_0^1 \log^2(gG^{-1}(x)/fF^{-1}(x))dx \right)^{\frac{1}{2}}}{\left| \int_0^1 \log(gG^{-1}(x)/fF^{-1}(x))dx \right|}$$

as n increases. Using Lemma 6.1, it is seen that (7.3) is unity if and only if H_o' is true, and exceeds unity otherwise. Thus, a test which rejects H_o' whenever (7.2) is "too large" will be consistent. The critical region can be approximated by using the limiting joint normality of the square of the numerator of (7.2) and the denominator without the absolute values (Theorem 4.1). The limiting distribution of (7.2) can be obtained using this information. It will be observed that the distribution will depend on F(x) through τ_1 so that for different distributions G(x) used in H_o, a different critical region will be appropriate. For the purpose of computing τ_1, it is not necessary to know μ and σ, and we can take μ to be 0 and σ to be unity. This follows from the fact that τ_1 is independent of the values of μ and σ.

To make a thorough comparison of (7.2) and Weiss's statistic by means of limiting power, the limiting distributions must be computed in some detail. A rough comparison can be obtained since both statistics converge to unity under H_o'', (7.2) converges to (7.3) in general and Weiss's statistic converges to

$$(7.3) \qquad \frac{\left| \int_0^1 \log(gG^{-1}(x)/fF^{-1}(x))dx \right|}{\left(\int_0^1 \log^2(gG^{-1}(x)/fF^{-1}(x))dx \right)^{\frac{1}{2}}}$$

as n increases. Using Lemma 6.1, it is seen that (7.3) is unity if and only if H_0' is true, and exceeds unity otherwise. Thus, a test which rejects H_0' whenever (7.2) is "too large" will be consistent. The critical region can be approximated by using the limiting joint normality of the square of the numerator of (7.2) and the denominator without the absolute values (Theorem 4.1). The limiting distribution of (7.2) can be obtained using this information. It will be observed that the distribution will depend on $F(x)$ through τ_1 so that for different distributions $G(x)$ used in H_0, a different critical region will be appropriate. For the purpose of computing τ_1, it is not necessary to know μ and σ, and we can take μ to be 0 and σ to be unity. This follows from the fact that τ_1 is independent of the values of μ and σ.

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(7.4) $$\left(\int_0^1 (gG^{-1}(x)/fF^{-1}(x))^2 dx\right)^{1/2} / \left(\int_0^1 (gG^{-1}(x)/fF^{-1}(x))dx\right).$$

If we take $(gG^{-1}(x)/fF^{-1}(x)) = x^k$ on $(0,1)$ we find that (7.3) exceeds (7.4) for $k = 1,2$ and the reverse for $k \geq 3$. Thus a first guess is that neither test is uniformly better than the other.

The scale parameter σ can be estimated by

(7.5) $$e^{\gamma}\left[\prod_{i=1}^{n-1} g(i)\Delta x_i\right]^{(1/n)} .$$

From Theorem 5.1, it is seen that (7.5) is a consistent estimate, and from Theorem 4.1 (7.5) is seen to have a log normal distribution with median σ. Approximate confidence intervals can be obtained from this limiting distribution.

It will be true in this case as well as the two sample case that the regularity conditions on the tails can be avoided by using truncated statistics.

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